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Advanced Statistical Physics - Problem set 11

Summer Terms 2022

Hand in: Hand in tasks marked with * to mailbox no. (43) inside ITP room 105b by Friday 24.06. at 9:15 am.

18. Coupling to a "massless" field *

2+2+2+2+2+2+2 Points

Consider an n-component vector field $\boldsymbol{m}(\boldsymbol{x})$ coupled to a scalar field $A(\boldsymbol{x})$, through the effective Hamiltonian

$$\beta\mathcal{H} = \int d^dx \left[\frac{K}{2}(\nabla\cdot\boldsymbol{m})^2 + \frac{t}{2}\boldsymbol{m}^2 + u(\boldsymbol{m}^2)^2 + e^2\boldsymbol{m}^2A^2 + \frac{L}{2}(\nabla A)^2\right],$$

with K, L, and u positive.

(a) Assume $\boldsymbol{m}(\boldsymbol{x}) = \overline{m}\ \hat{\boldsymbol{e}}_\ell$ and $A(\boldsymbol{x}) = 0$, and find the saddle point solution $\overline{m}$ for $t > 0$ and $t < 0$.

(b) Sketch the heat capacity $C = \partial^2 \ln Z/\partial t^2$ in the saddle point approximation, and discuss its singularity as $t \to 0$.

(c) Include fluctuations by setting

$$\begin{cases}\boldsymbol{m}(\boldsymbol{x}) = (\overline{m} + \phi_\ell(\boldsymbol{x}))\ \hat{\boldsymbol{e}}_\ell + \phi_t(\boldsymbol{x})\ \hat{\boldsymbol{e}}_t,\\ A(\boldsymbol{x}) = a(\boldsymbol{x}),\end{cases}$$

and expanding $\beta\mathcal{H}$ to quadratic order in ϕ and a.

Hint: after substituting the above in $\beta\mathcal{H}$, the linear terms vanish at the minimum and the second order terms give

$$\beta\mathcal{H}_2 = \int d^dx \left[\frac{K}{2}(\nabla\phi_\ell)^2 + \frac{t + 12\,u\,\overline{m}^2}{2}\,\phi_\ell^2\right] + \int d^dx \left[\frac{K}{2}(\nabla\phi_t)^2 + \frac{t + 4\,u\,\overline{m}^2}{2}\,\phi_t^2\right] + \int d^dx \left[\frac{L}{2}(\nabla a)^2 + \frac{2\,e^2\,\overline{m}^2}{2}a^2\right] + \mathcal{O}(\phi^3).$$

(d) Use your results from (c) to find the correlation lengths ξ_ℓ, and ξ_t, for the longitudinal and transverse components of ϕ, for $t > 0$ and $t < 0$.

(e) Find the correlation length ξ_a for the fluctuations of the scalar field a, for $t > 0$ and $t < 0$.

(f) Compute the correction to the saddle point free energy $-\ln(Z)/V$, from fluctuations. (You can leave the answer in the form of integrals involving ξ_ℓ, ξ_t, and ξ_a).

Hint: Perform a Fourier transform in the Hamiltonian $\beta\mathcal{H}_2$ to obtain Gaussian integrals over $D[\phi_t]$, $D[\phi_\ell]$, and $D[a]$. Then compute the Gaussian integrals to obtain an expression for the partition function Z. You do not need to perform the sums over momenta that you obtain after the Fourier transformation.

(g) Find the fluctuation corrections to the heat capacity in (b), again leaving the answer in the form of integrals.